

AD621563

**BRL**

REPORT NO. 658  
FEBRUARY 1948

0

CLEARINGHOUSE  
FOR FEDERAL SCIENTIFIC AND  
TECHNICAL INFORMATION

Hardcopy

Microfiche

\$1.00

\$0.50

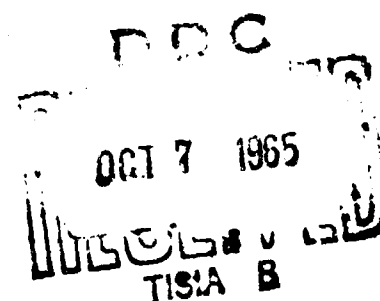
23 pp a

PROCESSING COPY

ARCHIVE COPY

THE ACCURACY OF DRAG MEASUREMENTS  
AS A FUNCTION OF NUMBER AND  
DISTRIBUTION OF TIMING STATIONS

B. G. Karpov



BALLISTIC RESEARCH LABORATORIES



ABERDEEN PROVING GROUND, MARYLAND

9

# **BALLISTIC RESEARCH LABORATORIES**

**REPORT NO. 658**

## **The Accuracy of Drag Measurements as a Function of Number and Distribution of Timing Stations**

**B. G. KARPOV**

**ORDNANCE RESEARCH AND DEVELOPMENT DIVISION  
OFFICE, CHIEF OF ORDNANCE  
PROJECT NO. TB3-01081**

**FEBRUARY 1948**

**ABERDEEN PROVING GROUND, MARYLAND**

---

---

## TABLE OF CONTENTS

	Page
ABSTRACT .....	3
INTRODUCTION .....	4
DERIVATION OF THE FORMULAE .....	5
ERRORS IN $v$ , $K_D$ , AND $dK_D/dv$ .....	8
OPTIMUM DISTRIBUTION OF TIMING STATIONS .....	11
THE ORDER OF TIME-DISTANCE POLYNOMIAL .....	12
USE OF POLYNOMIAL REPRESENTATION IN THE TRANSONIC REGION --	16
APPENDIX .....	18
DISTRIBUTION LIST .....	22

DUPLICATING BRANCH  
THE ORDNANCE SCHOOL  
ABERDEEN PROVING GROUND  
MARYLAND

---

---

**BALLISTIC RESEARCH LABORATORIES**  
**REPORT NO. 658**

Karpov/lrr  
Aberdeen Proving Ground, Md.  
18 November 1947

**THE ACCURACY OF DRAG MEASUREMENTS AS A FUNCTION  
OF NUMBER AND DISTRIBUTION OF TIMING STATIONS**

**ABSTRACT**

General formulae are given for the error to be expected in the drag coefficient of a projectile whose flight is observed over a given range containing  $n$  timing stations distributed in an arbitrary manner. It is assumed that the time-distance relation can be represented by a cubic polynomial of the form

$$t = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

It is further assumed that the mean errors in time and distance,  $e_t$  and  $e_z$  respectively, are independent and constant at each observing station.

Illustrative examples are included.

A proof of optimum distribution of timing stations for drag determination is given in the Appendix.

---

## INTRODUCTION

The Spark Range of the laboratory contains 25 spark stations which are arranged, in groups of five, over a distance of 280 feet. Every odd station is wired to record the time of passage of the projectile on the drum camera chronograph, 13 timing stations in all. In addition, stations 1, 7, 11, 13, 15, 19 and 25 are connected to six megacycle electronic counters which record six time intervals. The two timing systems serve as a check on each other. Distribution of station is illustrated in the schematic drawing below. The chronograph timing stations are marked by crosses; the counter stations are marked by dots. In both timing systems distribution of timing stations is symmetric with respect to the mid-point of the range.



With so many timing stations, distributed over a relatively short range, the drag coefficient can be determined with a high degree of accuracy with almost any reasonable symmetric distribution of the timing stations, but the actual distribution, when full range is used, appears to be near the optimum, i.e., leading to the smallest error in the drag.

However, in actual practice, not infrequently some timing stations misfire thus rendering the distribution assymmetric. Moreover, for special firings the range is sometimes shortened by cutting the last or the last two groups of stations. Under these circumstances it is desirable to know a priori how the accuracy of drag determination is affected by using a shortened range. If the accuracy is affected too adversely a redistribution of the remaining timing stations would be necessary.

The general problem of finding an optimum distribution of  $n$  timing stations, for drag determination, is of interest and frequently arises in practice. Work already has been done on this problem for cases of equal spacing of timing stations, and, more generally, for symmetric distribution of timing stations.<sup>1</sup> An analysis of error in the drag coefficient for an arbitrary distribution of the timing stations is a simple generalization of the symmetrical case and is a straight forward process. This generalization and the application of the results to the Spark Range are the subjects of this report.

1. J. L. Kelley; Unpublished work. Dr. Kelley considers the case of symmetric distribution of timing stations with time-distance relationship given by a quadratic polynomial.

Note: It is my understanding that Professor J. E. McShane has considered the more general problem of the optimum distribution of timing stations. Unfortunately, his work has never been published and was not available to me.

### DERIVATION OF THE FORMULAE

We shall assume that the time-distance relationship can be represented by a polynomial. Moreover, we shall assume that at each timing station the accidental errors in the time observations are characterized by a mean error  $e_t$ , and the accidental errors in the observation of distance are characterized by a mean error  $e_z$ . The respective mean errors are the same at each station and are independent.

We shall confine our discussion only to quadratic and cubic polynomials since there are both theoretical and practical reasons against using higher order polynomials for the spark range. Suppose, therefore, that the time-distance relation is given by

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 = t$$

where the origin of the  $z$  coordinates is arbitrary but is usually taken at the mid-point of the range, or at the average of the  $z$  coordinates.

With  $n$  timing stations we form normal equations in the usual manner, and, after eliminating  $a_0$ , we arrive at the following set

$$c_{11}a_1 + c_{12}a_2 + c_{13}a_3 = t_1$$

$$c_{22}a_2 + c_{23}a_3 = t_2$$

$$c_{33}a_3 = t_3$$

$$c_{ij} = c_{ji}$$

where

$$c_{11} = n \sum z^2 - (\sum z)^2$$

$$c_{12} = n \sum z^3 - \sum z \sum z^2$$

$$c_{13} = n \sum z^4 - \sum z \sum z^3$$

$$c_{22} = n \sum z^4 - (\sum z^2)^2$$

$$c_{23} = n \sum z^5 - \sum z^2 \sum z^3$$

$$c_{33} = n \sum z^6 - (\sum z^3)^2$$

$$t_1 = n \sum zt - \sum t \sum z \quad t_2 = n \sum z^2 t - \sum t \sum z^2 \quad t_3 = n \sum z^3 t - \sum t \sum z^3$$

We shall need the following auxiliary quantities

$$C_{12} = c_{11} c_{22} - c_{12}^2$$

$$C_{13} = c_{11} c_{23} - c_{12} c_{13}$$

$$C_{23} = c_{11} c_{33} - c_{13}^2$$

$$C_1 = c_{12} c_{23} - c_{13}^2$$

also

$$C_{33} = c_{13} c_{22} - c_{12} c_{23}$$

$$C_{44} = c_{22} c_{33} - c_{23}^2$$

$$C_2 = c_{12} c_{44} - c_{33}^2$$

Let us consider the timing error first. After solving the normal equations for the constants  $a_1$ ,  $a_2$ , and  $a_3$  we proceed to find errors  $e_1$ ,  $e_2$ , and  $e_3$  of these constants as functions of  $e_t$ , the timing error. The operations to be performed are the following:

$$\left(\frac{e_1}{e_t}\right)^2 = \sum_{k=1}^n \left(\frac{\partial a_1}{\partial t_k}\right)^2 \quad \text{where } i = 1, 2, 3 \\ k = 1, 2, \dots, n$$

the expression for each constant is partially differentiated with respect to each  $t$ , the result squared, and the squared expressions summed over all stations - a standard least square procedure.<sup>2</sup> The final results are given in the following table.

Table I. Error in constants  $a_1$ ,  $a_2$ , and  $a_3$  as functions of the timing error  $e_t$ , and the number and distribution of timing stations.

A. Arbitrary distribution of stations

The quantities tabulated are  $(e_1/e_t)^2$

Time-distance relation

Error	Quadratic	Cubic
$a_1$	$\frac{nc_{22}}{C_{12}}$	$\frac{nc_{22} C_{44}}{C_2}$
$a_2$	$\frac{nc_{11}}{C_{12}}$	$\frac{nc_{11} C_{23}}{C_1}$
$a_3$		$\frac{nc_{22} C_{12}}{C_2}$

B. Symmetric distribution of stations

Under this condition the sums of all odd powers of  $z$  vanish and the above equations assume much simpler form.

Error	Quadratic	Cubic
$a_1$	$\frac{1}{\sum z^2}$	$\frac{\sum z^6}{\sum z^2 \sum z^6 - (\sum z^4)^2}$
$a_2$	$\frac{n}{n \sum z^4 - (\sum z^2)^2}$	$\frac{n}{n \sum z^4 - (\sum z^2)^2}$
$a_3$		$\frac{\sum z^2}{\sum z^2 \sum z^6 - (\sum z^4)^2}$

2. For example, see W. E. Deming: Statistical Adjustment of Data, John Wiley & Sons Inc., 1943.

For symmetric distribution with equal spacing of stations the relations are further simplified. Let the number of timing stations be  $n = 2m + 1$ , always an odd number, the spacing  $h$ , and the overall range  $R$ . Then  $h = R/2m$  and the various sums of the coordinates can be written as:

$$\begin{aligned}\Sigma z^2 &= \frac{2R^2}{(2m)^2} \sum_{r=1}^m r^2 & \Sigma z^4 &= \frac{2R^4}{(2m)^4} \sum_{r=1}^m r^4 \\ \Sigma z^6 &= \frac{2R^6}{(2m)^6} \sum_{r=1}^m r^6\end{aligned}$$

Thus we write

### C. Symmetric and Equal spacing of stations

Error	Quadratic	Cubic
$a_1$	$\frac{(2m)^2}{2R^2} \frac{1}{\Sigma r^2}$	$\frac{(2m)^2}{2R^2} \frac{\Sigma r^6}{\Sigma r^2 \Sigma r^6 - (\Sigma r^4)^2}$
$a_2$	$\frac{(2m)^4}{4R^4} \frac{2m+1}{\frac{2m+1}{2} \Sigma r^4 - (\Sigma r^2)^2}$	the same as quadratic
$a_3$		$\frac{(2m)^6}{2R^6} \frac{\Sigma r^2}{\Sigma r^2 \Sigma r^6 - (\Sigma r^4)^2}$

all summations are to be performed from 1 to  $m$ .

It is to be noted that error in the velocity, which is proportional to  $e_1$ , varies inversely as the first power of the range; similarly error in the drag coefficient,  $e_2$ , varies inversely as the square of the range; and error in the derivative of  $K_D$ ,  $e_3$ , varies inversely as the cube of the range.

The products

$$\frac{e_1}{e_t} R = f_1, \quad \frac{e_2}{e_t} R^2 = f_2, \quad \text{and} \quad \frac{e_3}{e_t} R^3 = f_3$$

are independent of the range and depend only on the number of timing stations. These are tabulated below and show the sensitivity of various errors to variation in the number of timing stations:

Table II. Dependence of errors on the number of timing stations. Equal spacing of stations.

$2m + 1$	Quadratic	Quadratic and Cubic	Cubic		Quartic and Quintic
	$f_1$	$f_2$	$f_1$	$f_3$	$f_2$
6	1.28	4.28	3.80	16.87	—
7	1.13	3.93	3.07	14.70	16.67
9	1.03	3.66	2.71	13.66	14.18
11	0.96	3.41	2.46	12.72	12.78
13	0.89	3.22	2.37	12.04	11.81



The error in  $a_2$  increases nearly fourfold by using quartic or quintic polynomials. In order to attain the same accuracy in drag determination with quartic or quintic polynomials as with quadratic or cubic it would be necessary, therefore, to double the range. Increasing the number of the timing stations is much less effective means of reducing the error in  $a_2$  than lengthening the range.

Distance error. Distance error can be easily taken care of by the following device. We shall assume that distance error  $e_2$  can be expressed as equivalent timing error by the relation

$$e_t = \frac{e_2}{v}$$

where  $v$  is the velocity. We already have the expressions for errors in the constants  $a_1$ ,  $a_2$ , and  $a_3$  as functions of  $e_{t1}$  the true timing error. The error in these constants due to distance error, or equivalent timing error  $e_{t2}$ , can, therefore, be obtained at once by replacing  $e_{t1}$  by  $e_{t2}$ . The total timing error, therefore, the two errors being assumed independent, is

$$e_t^2 = e_{t1}^2 + e_{t2}^2$$

#### ERRORS IN $v$ , $K_D$ , AND $dK_D/dv$

Neglecting gravity, the equation of motion of the projectile can be written as

$$\frac{dv}{dz} = - \frac{v}{F_1} K_D$$

where  $F_1 = \frac{m}{\rho d^2}$ ,  $\rho$  being the air density,  $m$  is the mass, and  $d$  is the maximum diameter of the projectile.

If the time-distance relation is given by

$$t = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

then it can be easily verified that at  $z = 0$ , the following relations are true:

$$v_0 = \frac{1}{a_1} \quad K_D = F_1 \frac{2a_2}{a_1}$$

$$\frac{1}{F_1} \frac{dK_D}{dv} = 2a_2 - \frac{3a_3 a_1}{a_2}$$

Let the percentage mean errors in velocity, in  $K_D$ , and in its derivative be  $p_1$ ,  $p_2$ , and  $p_3$  respectively. Then

$$p_1 = 100 \frac{e_1}{a_1}$$

$$p_2 = 100 \left[ \left( \frac{e_2}{a_2} \right)^2 + \left( \frac{e_1}{a_1} \right)^2 \right]^{\frac{1}{2}}$$

For all practical purposes, however, the last term is much smaller than the first and with ample accuracy we can write

$$p_2 = 100 \frac{e_2}{a_2}$$

It can also be easily shown that the percentage error in the derivative of  $K_D$ ,  $p_3$ , is adequately given by

$$p_3 = 100 \frac{\frac{e_3}{\frac{2}{a_2} - a_3}}{1.5a_1}$$

The expressions for  $e_1$ ,  $e_2$ , and  $e_3$  already have been given. These are functions of the number and distribution of the timing stations, the length of the range, and the total timing error  $e_t$  and can be computed.

Let  $a$  be the velocity of sound,  $M = \frac{V}{a}$  the Mach number, then, using the definitions of constants  $a_1$ ,  $a_2$ , and  $a_3$  the percentage errors can be expressed as follows:

$$p_1 = 100 a M e_1$$

$$p_2 = 100 \frac{M}{K_D} 2F_1 a e_2$$

$$p_3 = 100 \frac{6 F_1^2 a}{K_D \frac{dK_D}{dM}} e_3$$

Consider the percentage error in  $K_D$ : It is proportional to  $M/K_D$  and since in supersonic velocities  $K_D$  usually decreases with increasing  $M$ ,  $p_2$  increases rather rapidly at higher velocities.  $p_2$  is also proportional to  $F_1$  or, by definition of  $F_1$ , it is proportional to the mass of the projectile and is inversely proportional to the air density, and to the square of the maximum diameter of the projectile. In addition,  $p_2$  is proportional to  $e_2$  which, in turn, depends upon the accuracy of the instrumentation as represented by the timing error  $e_1$  and the distance error  $e_2$  and upon the number and distribution of the timing stations. Finally,  $p_2$  depends on the order of the polynomial chosen to represent the time-distance relationship: the higher the order of the polynomial the greater the error in  $a_2$  ( $= e_2$ ) and, hence, in  $p_2$ . Therefore, in order to maintain the same accuracy in  $K_D$  with polynomial of higher order either the number of timing stations should be increased, or, more effectively, the range be increased. As an illustration, the percentage error in  $K_D$  has been computed for the case of a model of standard 155mm shell M101 which has been fired in the range at various Mach numbers. Although the firings were done only up to Mach number 2.5, the error curve was computed to  $M = 5.0$  by means of the empirical "Q function"

$$Q = \sqrt{1 + K_D M^2} = a + bM$$

For this model  $a = 0.9405$  and  $b = 0.1329$ .

Figure 1 shows separately the errors due to distance, to time, and the total error. It is apparent that at subsonic velocities the distance error is the more important; at higher velocities the error due to time predominates.

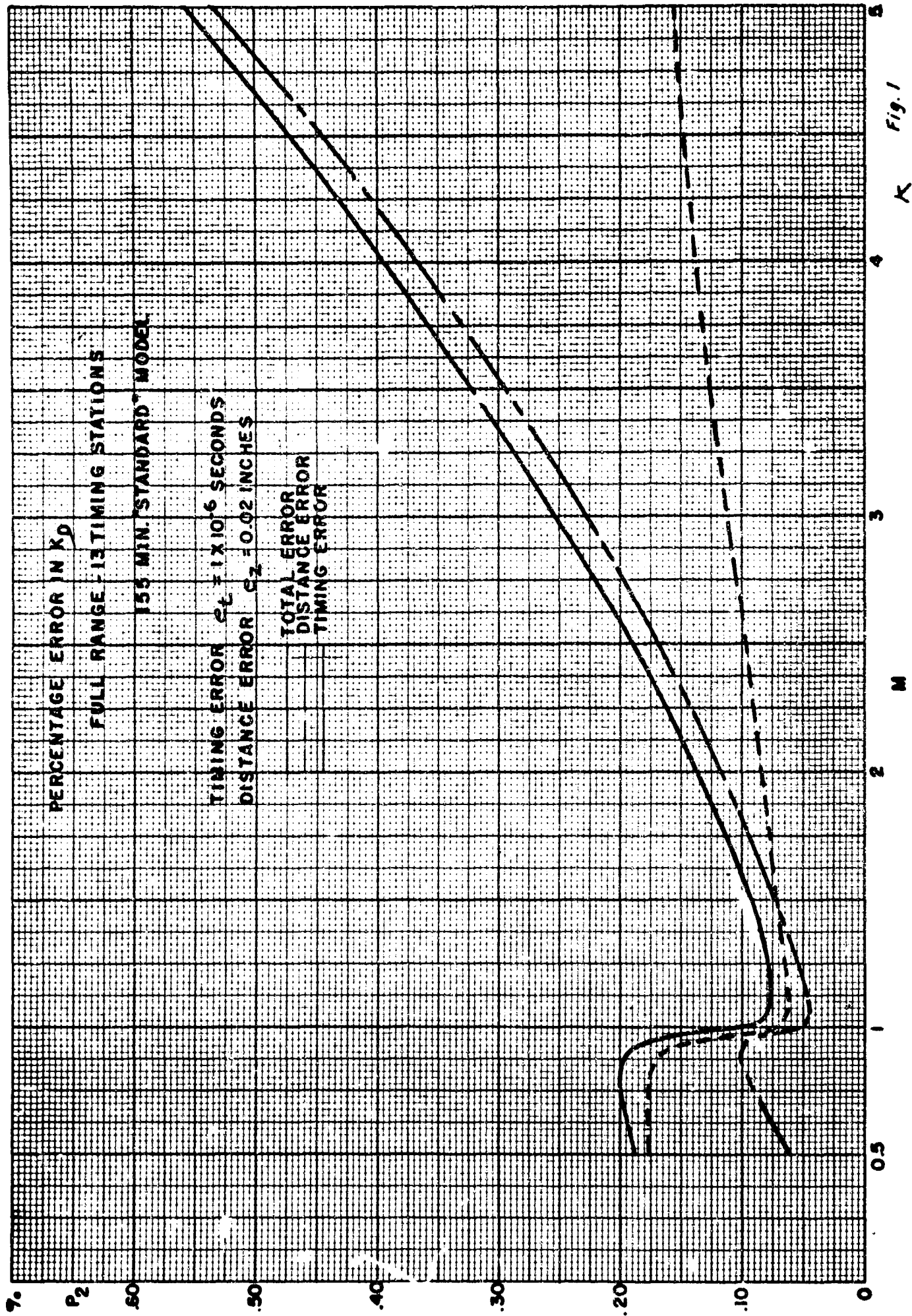


Fig. 1

### OPTIMUM DISTRIBUTION OF TIMING STATIONS

We have seen in the preceding section that the error in  $K_D$  is proportional to the error in  $a_2$ , which, in turn, with a given accuracy of observations, depends only upon the number and distribution of the timing stations. The question naturally arises, therefore, whether there is an optimum distribution of the  $n$  timing stations which will lead to the smallest error in the  $K_D$ .

For the case of symmetrical distribution of the timing stations when the time-distance relationship is given either by quadratic or cubic polynomials, Dr. H. G. Landau has shown that optimum distribution is attained if stations are grouped at each end of the range with one group in the middle. In fact, if the number of timing stations  $n$  is divisible by four, the best distribution calls for placing one half of the stations in the middle group, with one quarter of the stations at each end. The proof of this is to be found in the Appendix.

In general, if  $k$  be a factor such that  $n = 4k + m$  where  $m$  can be either 0, 1, 2, or 3, and the range be two units long, the following table shows the required optimum distributions.

Table III. Optimum distribution of timing stations

n	Number of stations at			$(\frac{e_2}{e_t})^2$
	$z = -1$	$z = 0$	$z = 1$	
$4k$	$k$	$2k$	$k$	$\frac{4}{n}$
$4k + 1$	$k$	$2k + 1$	$k$	$\frac{4n}{n^2 - 1}$
$4k + 2$	$k$ $k + 1$	$2k + 2$ $2k$	$k$ $k + 1$	$\frac{4n}{n^2 - 4}$
$4k + 3$	$k + 1$	$2k + 1$	$k + 1$	$\frac{4n}{n^2 - 1}$

To quote from the Appendix: "It should be pointed out that it will not be possible to fit a cubic in  $z$  with exactly these spacings of stations, since only three values of  $z$  are given. Because it is physically impossible to put more than one station at one position, the above spacing can only be approximated. . . .".

In the Spark Range the stations are arranged in groups of five, so we can use 11 timing stations for which the theoretical optimum distribution requires placing 3 stations at each end of the range and 5 in the middle. The following table compares  $e_2/e_t$  values for the theoretical optimum, for physically achievable distribution in the range with stations five feet apart, for our usual distribution of every odd station being a timing station, and also using seven electronic counter stations:

	No. of Timing Stations	$10^4 \times \frac{e_2}{e_t}$
Optimum	11	.295
Possible in the Range	11	.306
Usual distribution (drum camera)	13	.364
Electronic counters	7	.461

If the above figures are translated into the percentage error in  $K_D$ ,  $p_2$ , of model 155mm shell M101 for example, the effect on  $p_2$  of various distribution appears as follows:

Model 155mm shell M101	$p_2$ at $M = 2.0$
Optimum	0.109
Possible in the Range	0.113
Usual distribution (drum camera)	0.135
Electronic counters	0.171

Thus theoretical error, using only the electronic counters stations, is 50% larger than the optimum error attainable in the range. However, for various practical reasons it is undesirable to segregate the timing stations as indicated by the optimum distribution. Moreover, the whole error is so small that even seven counters provide ample accuracy.

The following additional few simple examples illustrate the effect of various distributions on  $e_2$ . Suppose the length of the range is 10 units. The table below gives various distributions and corresponding values of  $e_2/e_t$ .

Distribution	No. of stations	$e_2/e_t$
-5, $-2\frac{1}{2}$ , 0, 0, 0, $+2\frac{1}{2}$ , +5	7	.0360
-5, 0, 0, 0, +5	5	.0365
-5, -1, 0, +1, +5	5	.0375
-5, -2, 0, +2, +5	5	.0405
-5, -3, 0, +3, +5	5	.0453
-5, 0, 0, +5	4	.0400
One station misfires		
-5, x, 0, +2, +5	4	.0434 Quadratic
		.0490 Cubic

The table shows that it is possible to attain as good an accuracy with fewer stations properly located as with greater number placed less judiciously.

#### THE ORDER OF TIME-DISTANCE POLYNOMIAL

The order of the polynomial representing the time-distance relationship in a given range should be such that the next higher term should be less than the error in time measurement. This is a necessary condition. Thus if

$$t = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_m z^m$$

and the time is measured to the accuracy of  $10^{-6}$  seconds, the above condition requires that

$$\left| a_m z^m \right|_{\max} < 10^{-6}$$

If the  $z$ 's are reckoned from the mid point of the range,  $z_{\max} = \frac{R}{2}$ , therefore

$$a_m \left( \frac{R}{2} \right)^m < 10^{-6}$$

Given  $K_D$  vs  $M$ , the various coefficients can be computed. Thus

$$a_3 = \frac{K_D}{6F_1^2 a} \left( \frac{K_D}{M} - \frac{dK_D}{dM} \right)$$

and

$$a_4 = \frac{K_D^2 M}{24F_1^3 a} \left[ \frac{d^2 K_D}{dM^2} + \frac{1}{K_D} \left( \frac{K_D}{M} - \frac{dK_D}{dM} \right)^2 \right]$$

In the following table  $a_3$  and  $a_4$  are tabulated for various Mach numbers for the model 155mm shell M101 whose  $K_D$  vs  $M$  graph is given in Figure 2.

$M$	$a_3$	$a_4$
1.5	$4.582 \times 10^{-12}$	$-3.60 \times 10^{-16}$
2.5	1.688	-0.156
3.5	0.790	+0.032
4.5	0.439	+0.034

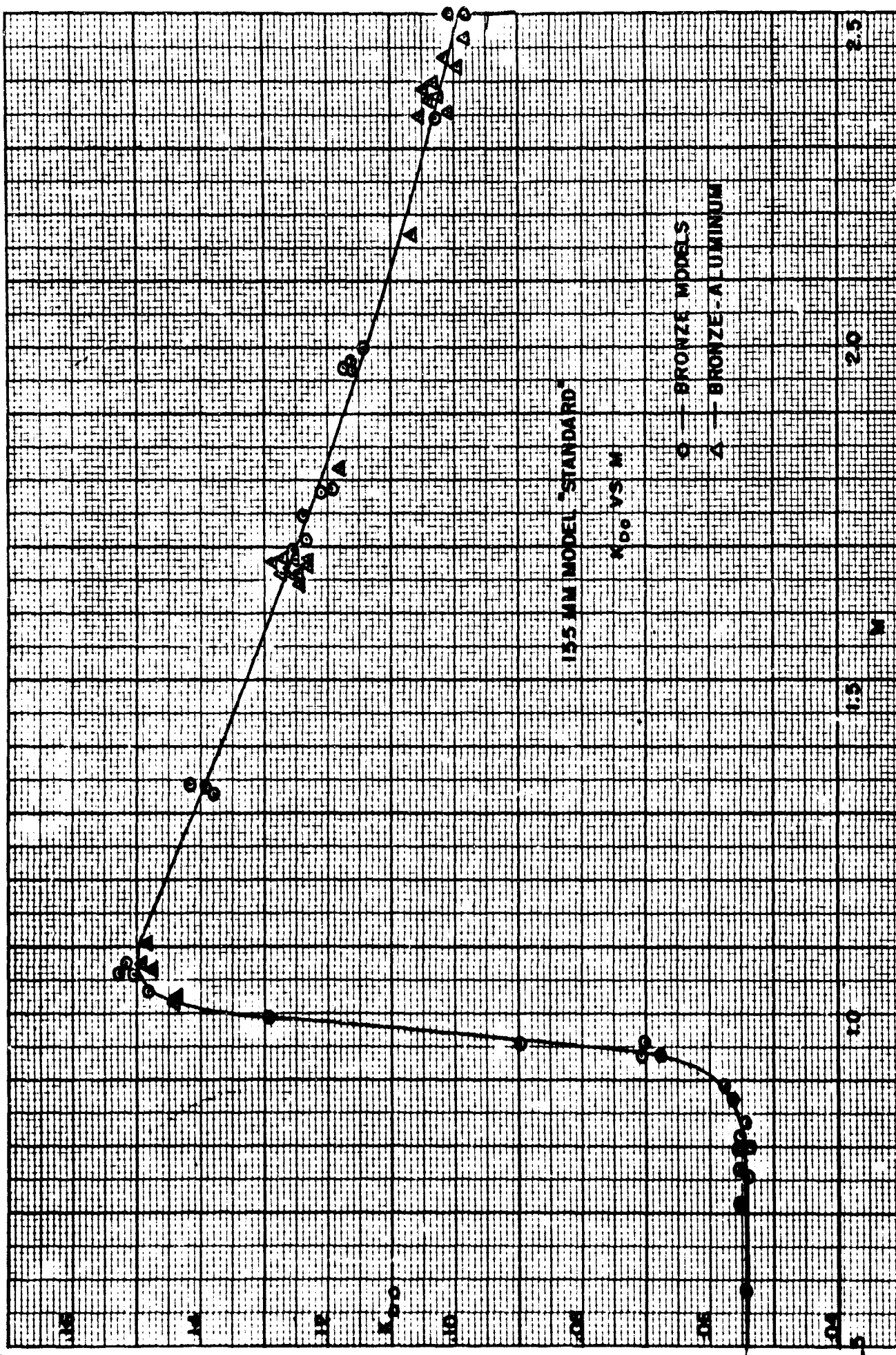
The  $K_D$  and its derivatives were computed from the  $Q$  function. The  $F_1$  for this shell is 750 feet. For estimating the order of the polynomial the above table can be used with sufficient accuracy for other shell provided the tabular values are multiplied by the ratio of 750 to  $F_1$  of the new shell, raised to appropriate power as indicated in the formulae.

In the following table  $a_3 \left( \frac{R}{2} \right)^3$  and  $a_4 \left( \frac{R}{2} \right)^4$  are tabulated for the spark range,  $R_1 = 280$  feet, and the transonic range,  $R_2 = 700$  feet. The transonic range of the laboratory, which is being built, will contain 25 photographic stations, arranged in groups of fives, over a distance of 700 feet. For this example it was assumed that 3-inch shells were to be fired in this range for which  $F_1 = 2660$  feet.

$M$	$a_3 \left( \frac{R}{2} \right)^3$		$a_4 \left( \frac{R}{2} \right)^4$	
	$R = 280$ 155mm model	$R = 700$ 3"	$R = 280$	$R = 700$
1.5	$12.55 \times 10^{-6}$	$15.59 \times 10^{-6}$	$-13.80 \times 10^{-8}$	$-4.84 \times 10^{-8}$
2.5	4.63	5.75	- 0.60	-0.21
3.5	2.16	2.68	+ 0.12	+0.04
4.5	1.20	1.49	+ 0.13	+0.05

On the basis of aforementioned criteria, therefore, in both spark and transonic ranges, the fourth power term can be safely omitted.

It is to be noted that although the cubic term is retained, the determination of  $\frac{dK_D}{dM}$  nevertheless, is very poor. This can be seen from the values of  $p_3$ , the percentage error, tabulated on page 15 for the model 155mm shell.



K Fig. 2

M	P <sub>3</sub>
1.5	71.2
2.5	133.8
3.5	281.5
4.5	457.9

Clearly, the derivative cannot be accurately determined from a single round.

It is of interest to see how well the time-distance relationship could be approximated by a polynomial if there were no observational errors. Advantage can be taken of the observed fact that in supersonic range of velocities, variation of the drag coefficient with Mach number is accurately represented by the Q function. Neglecting gravity, therefore, the equation of motion can be integrated explicitly. The result is:<sup>3</sup>

$$z = -At + B \ln \left( \frac{De^{-Ct} - 1}{D - 1} \right)^2$$

where A, B, C, and D are functions of M at z=0 and of the slope and intercept of the Q function of a particular type of projectile.

The computations were done for the cone-cylinder model, round 1730, which was fired at M=1.7. The z coordinates were computed by the above formula from the observed times thus freeing the z, t values from observational errors. Polynomials of various degrees were fitted to these values by least squares with origin of the coordinates being kept always at station 13, the midpoint of the range; for both symmetric and asymmetric distribution of timing stations. The following table contains, for each polynomial, the resulting K<sub>D</sub>, its percentage error p<sub>2</sub> computed from the residuals in the usual manner, and the mean error e<sub>0</sub>, in microseconds, of an equation of unit weight. The quality of fit can be judged from the size of e<sub>0</sub>.

Table IV

Distrib.	No. of stations	Quality of fit of z, t values by a polynomial			
			Quadratic	Cubic	Quartic
Symmetric	7	K <sub>D</sub>	.1250	.1250	.1227
		p <sub>2</sub>	.25	.14	.81
		e <sub>0</sub>	1.77	1.01	1.05
Asymmetric stations 1-19	3	K <sub>D</sub>	.1240	.1240	.1249
		p <sub>2</sub>	.05	.14	.95
		e <sub>0</sub>	.20	.24	.53
Asymmetric stations 7-25	6	K <sub>D</sub>	.1258	.1235	.1165
		p <sub>2</sub>	.40	.38	5.9
		e <sub>0</sub>	1.48	.66	3.05

3. I am indebted to Dr. A. C. Charters for this formula.



On the whole, at this Mach number, the cubic seem to give the best fit. however, quadratic could have been used equally well. Experience shows, however, that over the whole range of Mach numbers, and especially through the transonic range, cubic representation leads to more consistent results. It is to be noted that using the observed values of  $(z, t)$  the residuals in time, and hence the mean error, will usually be augmented by inherent inadequacy of the polynomial to fit the  $(z, t)$  function as shown in Table IV. Thus with the observed  $(z, t)$  values, using symmetric distribution and cubic polynomial, the results are:

$$K_D \quad .1251$$

$$p_2 \quad .29$$

$$e_o \quad 2.06$$

with the mean error,  $e_o$ , twice as large. However, one microsecond of this error is due to failure of the polynomial to fit the data, and only the remainder, therefore, is attributable to errors in time and distance.

#### USE OF POLYNOMIAL REPRESENTATION IN THE TRANSONIC REGION

Few remarks should be made concerning the use of polynomial fitting of  $(z, t)$  data in the transonic region where  $K_D$  varies rapidly with the Mach number.

It should be recalled that a cubic representation of the  $(z, t)$  data takes care of the linear part of the variation of  $K_D$  by the cubic term. In the spark range, the retardation of projectiles over 280 feet seldom exceeds .08 Mach numbers and is usually considerably less, so the linear approximation of variation of  $K_D$  vs  $M$  is adequately taken care of by the cubic term.

Another alternative sometimes is being advocated, namely, first to differentiate once, numerically, the  $(z, t)$  data, and represent the velocities by a quadratic in  $z$ . From the equation of motion, again neglecting gravity, we have

$$\ln v = F_1 \int K_D dz = C_0 + C_1 z + C_2 z^2$$

at  $z = 0$

$$\frac{d}{dz} \ln v = F_1 K_D = C_1$$

The above procedure, could perhaps be used successfully for projectiles with large retardation such as spheres or irregular fragments. For ordinary projectiles the reduction of data by the above method leads to somewhat larger errors in  $K_D$  than straight polynomial reduction.

The two methods designated I and II respectively have been applied to model firings of 155mm shell M101 with results tabulated on page 17. The values of  $K_D$  with corresponding percentage errors computed from residuals are tabulated for various  $M$ .

Table V  
Representation of  $z, t$  data by polynomial (I), and by method II.  
155mm shell M101

Rd.	M	Method I		Method II	
		$K_D$	$P_2$	$K_D$	$P_2$
1211	.797	.0590	.26	.0582	.30
1394	.934	.0655	.43	.0684	4.16
1207	.935	.0709	.10	.0709	.84
1208	.956	.0962	.18	.0925	2.01
1389	.993	.1299	.29	.1281	1.44
1206	1.017	.1444	.11	.1444	.91
1205	1.056	.1598	.18	.1581	1.43
1381	1.056	.1540	.19	.1537	1.01
1382	1.071	.1545	.09	.1535	.68
1318	1.965	.1187	.19	.1171	1.99
1312	1.999	.1164	.28	.1206	2.42

Although the differences in  $K_D$ 's computed by two methods are not excessive and are nonsystematic in character, the percentage errors by the second method appear to be about 10 times larger. Representation of  $(z, t)$  data by a cubic polynomial even in the transonic range must be considered quite satisfactory.

I wish to acknowledge my indebtedness to Mr. E. Dearden and Mr. K. G. Tadman of the British Branch for Theoretical Research, Fort Halstead, Kent, England.

*B. G. Karpov*

B. G. Karpov

# APPENDIX OPTIMUM DISTRIBUTION OF TIMING STATIONS FOR DRAG DETERMINATIONS

This appendix by H. G. Landau gives the solution of the following problem: To determine the drag coefficient for a projectile at the center of the Aerodynamics Range,  $n$  stations are placed symmetrically about the center of the range and the position,  $z$ , of each station, and time,  $t$ , when the projectile passes is determined. A quadratic or cubic in  $z$  is fitted to  $t$  by least squares, and the error in drag coefficient will be a minimum when the error in the coefficient of  $z^2$  is a minimum. How should the positions of the stations be chosen so as to minimize this error?

Since the square of the error in  $a_2$ , the coefficient of  $z^2$ , is proportional to the reciprocal of

$$n \sum_{i=1}^n z_i^4 - \left( \sum_{i=1}^n z_i^2 \right)^2$$

the problem is to maximize

$$S = n \sum_{i=1}^n z_i^4 - \left( \sum_{i=1}^n z_i^2 \right)^2$$

where the  $z_i$  are the distances of the stations from the center of the range and the scale is chosen so that the ends of the range are  $z = 1$  and  $z = -1$ .

$$\text{Now} \quad S = n \sum_{i=1}^n \left( z_i^2 - \frac{1}{n} \sum_{i=1}^n z_i^2 \right)^2$$

$$\text{Let} \quad x_i = z_i^2$$

$$\text{then} \quad S = n \sum_{i=1}^n \left( x_i - \frac{1}{n} \sum_{i=1}^n x_i \right)^2 \quad (1)$$

$$0 \leq x_i \leq 1 \quad (2)$$

and from the symmetry, every value of  $x_i \neq 0$  must occur an even number of times.

$S$  is, of course, the second moment of the  $x_i$  about their mean (times the constant,  $n$ ). The problem is to find the distribution of the  $x_i$  which maximizes  $S$  subject to (2). Physically, we must find the positions of  $n$  point-masses on a weightless bar of unit length which give the largest moment of inertia about the center of gravity. The answer is almost obvious intuitively: half the  $x_i$  must have the value 0 and the other half the value 1.

The proof can be given on the basis of the identity,

$$S = n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2 = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (x_i - x_j)^2 \quad (3)$$

which can be seen to be true by expanding

$$\begin{aligned}
 \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (x_i - x_j)^2 &= \frac{1}{2} \sum_{i=1}^n \left\{ \sum_{j=1}^n (x_i^2 - 2x_i x_j + x_j^2) \right\} \\
 &= \frac{1}{2} \sum_{i=1}^n \left\{ nx_i^2 - 2x_i \sum_{j=1}^n x_j + \sum_{j=1}^n x_j^2 \right\} \\
 &= \frac{1}{2} \left\{ n \sum_{i=1}^n x_i^2 - 2 \left( \sum_{i=1}^n x_i \right) \left( \sum_{j=1}^n x_j \right) + n \sum_{j=1}^n x_j^2 \right\} \\
 &= n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2.
 \end{aligned}$$

Let the  $x_i$  be numbered in order of increasing magnitude

$$0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1. \quad (4)$$

and suppose first that  $n$  is even. Then we can see that any distribution of the  $x_i$  gives  $S$  a smaller value than that given by the distribution with

$$x_i = 0 \text{ for } i \leq \frac{n}{2},$$

and

$$x_i = 1 \text{ for } i \geq \frac{n}{2} + 1. \quad (5)$$

For any  $i$ , the sum of the terms in  $S$  which include  $x_i$  are from (3)

$$S_i = \sum_{j=1}^n (x_i - x_j)^2$$

We show that if  $i \leq \frac{n}{2}$ , then  $S_i$  is increased by putting  $x_i = 0$ . Using (4)

$$S_i \leq (1-i) x_i^2 + \sum_{j=i+1}^n (x_i - x_j)^2. \quad (6)$$

Now

$$\begin{aligned}
 \sum_{j=i+1}^n x_j^2 &= \sum_{j=i+1}^n (x_j - x_i + x_i)^2 \\
 &= \sum_{j=i+1}^n (x_j - x_i)^2 + 2x_i \sum_{j=i+1}^n \left[ (x_j - x_i) + (n-i-1) x_i^2 \right];
 \end{aligned}$$

the second term here is positive and the last term is greater than  $(1-i) x_i^2$  because  $i \leq \frac{n}{2}$ . So that

$$\sum_{j=i+1}^n x_j^2 \geq \sum_{j=i+1}^n (x_j - x_i)^2 + (1-i) x_i^2.$$

which, using (6) gives,

$$S_1 \leq \sum_{j=1}^n x_j^2 \leq \sum_{j=1}^{i-1} x_j^2 + \sum_{j=i+1}^n x_j^2$$

the value of  $S_1$  for  $x_1 = 0$ .

In exactly the same way it can be seen that if  $i \geq \frac{n}{2} + 1$ ,  $S_1$  is increased by putting  $x_i = 1$ .

Hence it follows that the maximum value of  $S$  is given by the distribution (5).

If  $n$  is a multiple of 4,  $n = 4k$ , this gives the answer immediately: half of the  $z_i$  have the value 0, one-quarter will be +1 and one quarter will be -1.

The modification for  $n$  not a multiple of four can be seen as follows:

a.  $n$  odd. Just as before we must have  $x_1 = 0$  for  $i \leq \frac{n-1}{2}$ , and  $x_i = 1$  for  $i \geq \frac{n+3}{2}$ . The value of  $x_{\frac{n+1}{2}}$  is not yet determined.

The sum of the terms which include  $x_{\frac{n+1}{2}}$  is

$$\begin{aligned} S_{\frac{n+1}{2}} &= \frac{n-1}{2} \left[ x_{\frac{n+1}{2}}^2 + (1 - x_{\frac{n+1}{2}})^2 \right] \\ &= \frac{n-1}{2} \left[ 1 - 2x_{\frac{n+1}{2}}(1 - x_{\frac{n+1}{2}}) \right] \leq \frac{n-1}{2} \end{aligned}$$

The equality is reached for either  $x_{\frac{n+1}{2}} = 0$  or 1.

For  $n = 4k + 1$ , the symmetry condition requires putting  $x_{\frac{n+1}{2}} = 0$ , and for  $n = 4k + 3$  symmetry requires  $x_{\frac{n+1}{2}} = 1$ .

b.  $n = 4k + 2$ . In this case the distribution (5) would not satisfy the symmetry condition. We know that for  $S$  to be a maximum we must have

$$x_i = 0 \text{ for } i \leq \frac{n-2}{2}, \text{ and } x_i = 1 \text{ for } i \geq \frac{n+4}{2},$$

and symmetry requires  $x_{\frac{n}{2}} = x_{\frac{n+2}{2}}$ . Then just as in (a) above, it follows that these two  $x$ 's either both have the value 0 or both equal 1.

The results are summarized in the following table.

n	Number of values of z at			S
	z = -1	z = 0	z = 1	
4k	k	2k	k	$\frac{n^2}{4}$
4k + 1	k	2k + 1	k	$\frac{n^2 - 1}{4}$
4k + 2	k	2k + 2	k	$\frac{n^2 - 1}{4}$
	k + 1	2k	k + 1	
4k + 3	k + 1	2k + 1	k + 1	$\frac{n^2 - 1}{4}$

It should be pointed out that it will not be possible to fit a cubic in  $z$  with exactly these spacings of stations, since only three values of  $z$  are given. Because it is physically impossible to put more than one station at one position, the above spacing can only be approximated, and the coefficient of  $z^3$  can then be determined, but its error will be large. If this coefficient is desired with any accuracy the problem must be reconsidered, taking this into account.

---

AD-621563 *Filed*

WDM FORM A (13 MAR 47)

Karpov, B.

DIVISION: Ordnance and Armament (22)

SECTION: Ballistics (12)

CROSS REFERENCES: Projectiles - Drag (75415.6);  
Missile tracking (62019)

ATI- 19014

ORIG. AGENCY NUMBER  
BRLR-658

REVISION

AUTHOR(S)

AMER. TITLE: The accuracy of drag measurements as a function of number and distribution of timing stations

FORG'N. TITLE:

ORIGINATING AGENCY: Aberdeen Proving Ground, Ballistic Research Lab., Md.

TRANSLATION:

COUNTRY	LANGUAGE	FORG'N. CLASS	U. S. CLASS	DATE	PAGES	ILLUS.	FEATURES
U.S.	Eng.		Restr.	Feb'48	24	4	tables, graphs

**ABSTRACT**

General formulas are given for the error to be expected in the drag coefficient of a projectile whose flight is observed over a given range containing arbitrarily distributed timing stations. It is assumed that the time distance relation can be represented by a cubic polynomial of the form  $t = a_0 + a_1s + a_2s^2 + a_3s^3$ , and that the mean errors in time and distance are independent and constant at each observing station. In addition, a proof of optimum distribution of timing stations for drag determination is given.

*Ltd Approval Must Be Obtained from ORDA WASH D C*

T-2, HQ., AIR MATERIEL COMMAND

**AIR TECHNICAL INDEX  
RESTRICTED**

WRIGHT FIELD, OHIO, USAAF

WH-D-31 MAR 47 22,000

*P 19/01*

*NOV 1953*

*(23) Drag*